

COMPLETE k -CURVATURE HOMOGENEOUS PSEUDO-RIEMANNIAN MANIFOLDS

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ABSTRACT. For $k \geq 2$, we exhibit complete k -curvature homogeneous neutral signature pseudo-Riemannian manifolds which are not locally affine homogeneous (and hence not locally homogeneous). All the local scalar Weyl invariants of these manifolds vanish. These manifolds are Ricci flat, Osserman, and Ivanov-Petrova.

1. INTRODUCTION

We consider a pair $\mathcal{M} := (M, g_M)$ where g_M is a pseudo-Riemannian metric of signature (p, q) on a smooth manifold M of dimension $m := p + q$. Let R be the associated Riemann curvature tensor and let $\nabla^k R$ denote the k^{th} covariant derivative of the curvature tensor. We say that \mathcal{M} is k -curvature homogeneous if given any two points $P, Q \in M$, there exists an isomorphism $\phi_{P,Q}$ from $T_P M$ to $T_Q M$ so that

$$\phi^* g_Q = g_P, \quad \phi^* R_Q = R_P, \quad \dots, \quad \phi^* \nabla^k R_Q = \nabla^k R_P.$$

This means that the metric, curvature tensor, and covariant derivatives up to order k of the curvature tensor “look the same” at each point.

There is an equivalent algebraic formalism. Consider

$$\mathcal{U}_m^k := (V, g, A^0, A^1, \dots, A^k)$$

where g is an inner product on a m dimensional real vector space V and where $A^i \in \otimes^{4+i} V^*$ for $0 \leq i \leq k$. We say that \mathcal{U}_m^k is a k -model for \mathcal{M} if for every point $P \in M$, there is an isomorphism $\phi_P : T_P M \rightarrow V$ so that

$$\phi_P^* g = g_P, \quad \phi_P^* A^0 = R_P, \quad \dots, \quad \phi_P^* A^k = \nabla^k R_P.$$

If \mathcal{M} is k -curvature homogeneous, then $\mathcal{U}_{m,P}^k := (T_P M, g_P, R_P, \dots, \nabla^k R_P)$ is a k -model for \mathcal{M} for any point $P \in M$; conversely, if \mathcal{M} admits a k -model, then \mathcal{M} is k -curvature homogeneous.

There are a number of important results in this area in the Riemannian setting ($p = 0$). Takagi [29] was the first to exhibit 0-curvature homogeneous manifolds which are not locally homogeneous; his examples were non compact. Subsequently, compact examples were exhibited by Ferus, Karcher, and Münzer [8]. Tomassini [30] studied principal fiber bundles with 1 dimensional fiber over a 0-curvature homogeneous base. Other examples may be found in [19, 20, 31, 34]. Tsukada [32] classified 0-curvature homogeneous hypersurfaces of dimension $m \geq 4$ in complete and simply connected Riemannian space forms; the case $m = 3$ was subsequently treated by Calvaruso, Marinosci, and Perrone [6]. Kowalski and Prüfer [18] exhibited 4 dimensional algebraic curvature tensors which are not realizable by any 0-curvature homogeneous space.

Key words and phrases. affine k -curvature homogeneous, Ivanov-Petrova manifold, k -curvature homogeneous, locally affine homogeneous, Osserman manifold, Ricci flat.
2000 *Mathematics Subject Classification.* 53B20.

Scalar invariants can be obtained by using the Weyl calculus to contract indices in pairs in a polynomial expression involving the curvature and its higher covariant derivatives. For example, the scalar curvature is defined by setting

$$\tau := \sum_{ijkl} g^{ij} g^{kl} R_{iklj} .$$

Clearly if \mathcal{M} is locally homogeneous, then all such scalar invariants are necessarily constant.

We summarize some important results in this field in the Riemannian setting:

Theorem 1.1. *Let \mathcal{M} be a Riemannian manifold. Then:*

- (1) **(Tricerri and Vanhecke [33])** *If \mathcal{M} is modeled on an irreducible Riemannian symmetric space \mathcal{N} , then \mathcal{M} is locally symmetric and hence locally isometric to \mathcal{N} .*
- (2) **(Singer [26])** *There exists an integer k_m so that if \mathcal{M} is a complete simply connected manifold of dimension m which is k_m -curvature homogeneous, then \mathcal{M} is homogeneous.*
- (3) **(Prüfer, Tricerri, and Vanhecke [24])** *If all local scalar Weyl invariants up to order $\frac{1}{2}m(m-1)$ are constant on a Riemannian manifold \mathcal{M} , then \mathcal{M} is locally homogeneous and \mathcal{M} is determined up to local isometry by these invariants.*

We remark that Cahen et. al. [5] used a classification result of Berger to show that if \mathcal{M} is a Lorentzian ($p = 1$) manifold which is modeled on an irreducible Lorentzian symmetric space, then \mathcal{M} has constant sectional curvature. Thus Assertion (1) has a natural, and even stronger, extension to the Lorentzian setting.

Singer established the bound $k_m < \frac{1}{2}m(m-1)$. Bounds of $3m-5$ and $\frac{3}{2}m-1$ for k_m have been established by Yamato [35] and Gromov [14]. In the low dimensional setting, K. Sekigawa, H. Suga, and L. Vanhecke [27, 28] showed that $k_3 = k_4 = 1$. We refer to the discussion in Boeckx, L. Vanhecke, and O. Kowalski [2] for further details concerning k -curvature homogeneous manifolds in the Riemannian setting.

Theorem 1.1 (2) extends to the pseudo-Riemannian setting:

Theorem 1.2. (Podesta and Spiro [25]) *There exists an integer $k_{p,q}$ so that if (M, g) is a complete simply connected pseudo-Riemannian manifold of signature (p, q) which is $k_{p,q}$ -curvature homogeneous, then (M, g) is homogeneous.*

B. Opozda [22] has established an analogue of this result in the affine setting.

In the Lorentzian setting, examples of curvature homogeneous manifolds which are not locally homogeneous were constructed by Cahen et. al. [5]. Subsequently, 1-curvature homogeneous manifolds which are not locally homogeneous have been constructed by Bueken and Vanhecke [4]; we also refer to related work of Bueken and Djorić [3]. These examples are important since they show the results of [27, 28] do not extend to the indefinite setting. Pravda, Pravdová, Coley, and Milson [23] exhibited Lorentz manifolds all of whose scalar Weyl invariants vanish and which are not locally homogeneous; thus Theorem 1.1 (3) is false in this setting.

Not as much is known in the higher signature context. The authors [12] exhibited a family of complete 1-curvature homogeneous pseudo-Riemannian manifolds of signature $(r+1, r+1)$ on \mathbb{R}^{2r+2} for $r \geq 2$ which were 0-modeled on an irreducible symmetric space and which were not 2-curvature homogeneous (and hence not homogeneous); two other families of 0-curvature pseudo-Riemannian manifolds were also exhibited that are 0-modeled on irreducible symmetric spaces. Thus Theorem 1.1 (1) fails in the higher signature setting. We also refer to [13] for other examples of 0-curvature homogeneous pseudo-Riemannian manifolds.

Let $k = p + 2 \geq 2$ be given. In this paper, we will exhibit a family of complete neutral signature metrics $g_{2p+6, f}$ on \mathbb{R}^{2p+6} which are k -curvature homogeneous but not locally homogeneous for generic values of f . We shall be defining a number of

tensors. To simplify the discussion, we shall only give the non-zero entries in these tensors up to the usual \mathbb{Z}_2 symmetries.

Introduce coordinates $(x, y, z_0, \dots, z_p, \bar{x}, \bar{y}, \bar{z}_0, \dots, \bar{z}_p)$ on \mathbb{R}^{2p+6} . Let $f = f(y)$ be a smooth function on \mathbb{R} . Let $\mathcal{M}_{2p+6,f} := (\mathbb{R}^{2p+6}, g_{2p+6,f})$ be the pseudo-Riemannian manifold of balanced (i.e. neutral) signature $(p+3, p+3)$ where:

$$\begin{aligned} F_{2p+6,f}(y, \vec{z}) &:= f(y) + yz_0 + y^2 z_1 + \dots + y^{p+1} z_p, \\ g_{2p+6,f}(\partial_{z_i}, \partial_{\bar{z}_j}) &= \delta_{ij}, \quad g_{2p+6,f}(\partial_x, \partial_{\bar{x}}) = 1, \\ g_{2p+6,f}(\partial_y, \partial_{\bar{y}}) &= 1, \quad \text{and} \quad g_{2p+6,f}(\partial_x, \partial_x) = -2F_{2p+6,f}(y, \vec{z}). \end{aligned}$$

Choose a basis \mathcal{B} for \mathbb{R}^{2p+6} of the form

$$\mathcal{B} := \{X, Y, Z_0, \dots, Z_p, \bar{X}, \bar{Y}, \bar{Z}_0, \dots, \bar{Z}_p\}.$$

Consider the models $\mathcal{U}_{2p+6}^i := (\mathbb{R}^{2p+6}, g_{2p+6}, A_{2p+6}^0, \dots, A_{2p+6}^i)$ for $0 \leq i \leq p+2$ where the inner product g_{2p+6} and the tensors $A_{2p+6}^i \in \otimes^{4+i}(\mathbb{R}^{2p+6})^*$ have non-zero components

$$\begin{aligned} (1.a) \quad & g_{2p+6}(X, \bar{X}) = g_{2p+6}(Y, \bar{Y}) = g_{2p+6}(Z_i, \bar{Z}_i) = 1, \quad A_{2p+6}^0(X, Y, Z_0, X) = 1, \\ & A_{2p+6}^1(X, Y, Z_1, X; Y) = A_{2p+6}^1(X, Y, Y, X; Z_1) = 1, \\ & A_{2p+6}^2(X, Y, Z_2, X; Y, Y) = A_{2p+6}^2(X, Y, Y, X; Z_2, Y) \\ & = A_{2p+6}^2(X, Y, Y, X; Y, Z_2) = 1, \dots \\ & A_{2p+6}^p(X, Y, Z_p, X; Y, \dots, Y) = A_{2p+6}^p(X, Y, Y, X; Z_p, Y, \dots, Y) \\ & = \dots = A_{2p+6}^p(X, Y, Y, X; Y, \dots, Y, Z_p) = 1, \\ & A_{2p+6}^{p+1}(X, Y, Y, X; Y, \dots, Y) = 1, \quad \text{and} \quad A_{2p+6}^{p+2}(X, Y, Y, X; Y, \dots, Y) = 1. \end{aligned}$$

Theorem 1.3.

- (1) All geodesics in $\mathcal{M}_{2p+6,f}$ extend for infinite time.
- (2) $\exp_P : T_P \mathbb{R}^{2p+6} \rightarrow \mathbb{R}^{2p+6}$ is a diffeomorphism for any $P \in \mathbb{R}^{2p+6}$.
- (3) \mathcal{U}_{2p+6}^p is a p -model for $\mathcal{M}_{2p+6,f}$.
- (4) If $f^{(p+3)} > 0$ and $f^{(p+4)} > 0$, then \mathcal{U}_{2p+6}^{p+2} is a $p+2$ -model for $\mathcal{M}_{2p+6,f}$.

It is convenient to work in the affine setting. Let

$$\mathcal{R}(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$$

be the curvature operator defined by a torsion free connection ∇ on the tangent bundle of a smooth manifold M . Following Opzoda [21], we say that (M, ∇) is affine k -curvature homogeneous if given any two points P and Q of M , there is a linear isomorphism $\phi : T_P M \rightarrow T_Q M$ so that $\phi^* \nabla^i \mathcal{R}_Q = \nabla^i \mathcal{R}_P$ for $0 \leq i \leq k$. Taking ∇ to be the Levi-Civita connection of a pseudo-Riemannian metric then yields that any k -curvature homogeneous manifold is necessarily affine k -curvature homogeneous by simply forgetting the requirement that ϕ be an isometry; there is no metric present in the affine setting.

We say that (M, ∇) is *locally affine homogeneous* if given any points P and Q in M , there is a diffeomorphism Φ from a neighborhood of P to a neighborhood of Q so that $\Phi(P) = Q$ and so that $\Phi^* \nabla = \nabla$. If (M, ∇) is locally affine homogeneous, necessarily (M, ∇) is affine k -curvature homogeneous for any k . Examples of 2-curvature homogeneous affine manifolds which are not locally affine homogeneous are known; we refer to the discussion in [9, 15, 16, 17, 21] for this and related results.

We will show that all the scalar Weyl invariants of $\mathcal{M}_{2p+6,f}$ vanish; these manifolds provide additional examples showing Theorem 1.1 (3) fails in the higher signature setting. To show that $\mathcal{M}_{2p+6,f}$ is not locally homogeneous, we must define a suitable invariant. We assume $f^{(p+4)} > 0$ and set

$$\alpha_{2p+6,f} := \frac{f^{(p+3)} f^{(p+5)}}{f^{(p+4)} f^{(p+4)}}.$$

Let ∇ be the Levi-Civita connection of $g_{2p+6,f}$. We will show that $\alpha_{2p+6,f}$ is a local affine invariant of $(\mathbb{R}^{2p+6}, \nabla)$; it is not of Weyl type. For generic f , the zero set of the derivative $\alpha'_{2p+6,f}$ is discrete and hence $\alpha_{2p+6,f}$ is not constant on any open set; thus, for generic f , $\mathcal{M}_{2p+6,f}$ is not locally affine homogeneous and hence not locally homogeneous; furthermore, the scalar Weyl invariants do not determine $\mathcal{M}_{2p+6,f}$ up to local isometry.

Theorem 1.4. *Assume that $f^{(p+3)} > 0$ and that $f^{(p+4)} > 0$. Then:*

- (1) *All scalar Weyl invariants of $\mathcal{M}_{2p+6,f}$ vanish.*
- (2) *If $\mathcal{M}_{2p+6,f}$ is affine $p+3$ curvature homogeneous, then $\alpha_{2p+6,f}$ is constant.*
- (3) *If ϕ is a local diffeomorphism of $\mathcal{M}_{2p+6,f}$ such that $\phi^*\nabla = \nabla$, then we have that $\phi^*\alpha_{2p+6,f} = \alpha_{2p+6,f}$.*
- (4) *If $\alpha_{2p+6,f}$ is non-constant, then $\mathcal{M}_{2p+6,f}$ is not locally affine curvature homogeneous.*

This theorem provides a lower bound for Singer's constant in the neutral setting by showing that if $p \geq 0$, then $k_{p+3,p+3} \geq p+3$ since f can be chosen so $\mathcal{M}_{2p+6,f}$ is $p+2$ -curvature homogeneous but not locally $p+3$ -affine curvature homogeneous. By taking suitable product structures and by using the 3 dimensional Lorentzian examples [4] which are 1-curvature homogeneous but not locally homogeneous, one may establish the lower bound

$$k_{p,q} \geq \min(p, q).$$

This also establishes a corresponding lower bound in the affine setting for the Opozda constant [22].

There are two special cases which are important. Set

$$\mathcal{M}_{2p+6}^1 := \mathcal{M}_{2p+6,e^y} \quad \text{and} \quad \mathcal{M}_{2p+6}^2 := \mathcal{M}_{2p+6,e^y+e^{2y}}.$$

Theorem 1.5.

- (1) \mathcal{M}_{2p+6}^1 is a homogeneous space.
- (2) \mathcal{M}_{2p+6}^2 is $2p+2$ -modeled on \mathcal{M}_{2p+6}^1 .
- (3) \mathcal{M}_{2p+6}^2 is not locally $p+3$ -affine curvature homogeneous.

The Jacobi operator is the self-adjoint operator characterized by the property $g(J(X)Y, Z) = R(Y, X, X, Z)$. One says that \mathcal{M} is *nilpotent Osserman* if 0 is the only eigenvalue of the Jacobi operator $J(X)$ for any tangent vector X . If $\{e_1, e_2\}$ is an oriented orthonormal basis for a non-degenerate 2-plane π , then the skew-symmetric endomorphism $\mathcal{R}(\pi) := \mathcal{R}(e_1, e_2)$ is independent of the particular basis chosen. One says that \mathcal{M} is *nilpotent Ivanov-Petrova* if 0 is the only eigenvalue of $\mathcal{R}(\pi)$ for any such π . We refer to [10, 11] for a further discussion of these operators in this context.

Theorem 1.6. *$\mathcal{M}_{2p+6,f}$ is Ricci flat, nilpotent Osserman, and nilpotent Ivanov-Petrova.*

Theorem 1.1 (1) fails in this setting. We refer to [12] for a further discussion of this phenomena and here content ourselves with showing:

Theorem 1.7. *Assume that $f^{(3)} > 0$ and $f^{(4)} > 0$. Then $\mathcal{M}_{6,f}$ is a 6 dimensional neutral signature manifold which is 2-curvature homogeneous, which is complete, which is modeled on an irreducible neutral signature symmetric space, all of whose local scalar Weyl invariants vanish identically, and which is not affine 3-curvature homogeneous for generic f .*

There is a 4 dimensional example $\mathcal{M}_{4,f} := (\mathbb{R}^4, g_{4,f})$ where

$$g_{4,f}(\partial_x, \partial_x) = -2f(y) \quad \text{and} \quad g_{4,f}(\partial_x, \partial_{\bar{x}}) = g_{4,f}(\partial_y, \partial_{\bar{y}}) = 1.$$

This example is defined, at least in a formal sense, by setting $p = -1$ in the discussion given above. Assume $f^{(2)} > 0$ and $f^{(3)} > 0$. Dunn [7] showed that $\mathcal{M}_{4,f}$ is a 1-curvature homogeneous complete manifold which is 0-modeled on an irreducible symmetric space and which is not locally homogeneous for generic f .

The remainder of this paper is devoted to the proof Theorems 1.3-1.7. In Section 2, we determine the Christoffel symbols of the connection ∇ relative to the coordinate frame and establish Assertions (1) and (2) of Theorem 1.3. In Section 3, we compute the curvature of the metric $g_{2p+6,f}$; Theorem 1.4 (1) and Theorem 1.6 follow from this computation. In Section 4, we prove Assertions (3) and (4) of Theorem 1.3. In Section 5, we complete the proof of Theorem 1.4; Theorem 1.5 follows as a scholium to these computations. We conclude the paper in Section 6 with the proof of Theorem 1.7.

2. THE GEODESICS OF $\mathcal{M}_{2p+6,f}$

The non-zero Christoffel symbols of the first and second kinds are given by:

$$\begin{aligned} g_{2p+6,f}(\nabla_{\partial_x} \partial_y, \partial_x) &= g_{2p+6,f}(\nabla_{\partial_y} \partial_x, \partial_x) = -g_{2p+6,f}(\nabla_{\partial_x} \partial_x, \partial_y) \\ &= -\partial_y F_{2p+6,f}, \\ g_{2p+6,f}(\nabla_{\partial_{z_i}} \partial_x, \partial_x) &= g_{2p+6,f}(\nabla_{\partial_x} \partial_{z_i}, \partial_x) = -g_{2p+6,f}(\nabla_{\partial_x} \partial_x, \partial_{z_i}) \\ &= -y^{i+1}, \end{aligned}$$

and by

$$\begin{aligned} \nabla_{\partial_x} \partial_y &= \nabla_{\partial_y} \partial_x = -(\partial_y F_{2p+6,f}) \partial_{\bar{x}}, \\ \nabla_{\partial_x} \partial_x &= (\partial_y F_{2p+6,f}) \partial_{\bar{y}} + \sum_i y^{i+1} \partial_{\bar{z}_i}, \\ \nabla_{\partial_x} \partial_{z_i} &= \nabla_{\partial_{z_i}} \partial_x = -y^{i+1} \partial_{\bar{x}}. \end{aligned}$$

This exhibits a crucial feature of this metric:

$$(2.a) \quad \nabla\{\partial_x, \partial_y, \partial_{z_i}\} \in \text{Span}\{\partial_{\bar{x}}, \partial_{\bar{y}}, \partial_{\bar{z}_i}\}, \quad \text{and} \quad \nabla\{\partial_{\bar{x}}, \partial_{\bar{y}}, \partial_{\bar{z}_i}\} = \{0\}.$$

Assertions (1) and (2) of Theorem 1.3 will follow from the following technical Lemma by setting:

$$\begin{aligned} u_1 &= x, & u_2 &= y, & u_3 &= z_0, & \dots, & u_{p+3} &= z_p, \\ u_{p+4} &= \bar{x}, & u_{p+5} &= \bar{y}, & u_{p+6} &= \bar{z}_0, & \dots, & u_{2p+6} &= \bar{z}_p. \end{aligned}$$

Lemma 2.1. *Let (u_1, \dots, u_n) be coordinates on \mathbb{R}^n . Let g be a pseudo-Riemannian metric on \mathbb{R}^n so that $\nabla_{\partial_{u_a}} \partial_{u_b} = \sum_{a,b < c} \Gamma_{ab}^c(u_1, \dots, u_{c-1}) \partial_{u_c}$. Then:*

- (1) (\mathbb{R}^n, g) is a complete pseudo-Riemannian manifold.
- (2) $\exp_P : T_P \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a diffeomorphism for all P in \mathbb{R}^n .

Proof. We shall adopt the notational convention that the empty sum is 0. Let $\gamma(t) = (u_1(t), \dots, u_n(t))$ be a curve in \mathbb{R}^n ; γ is a geodesic if and only

$$\ddot{u}_c(t) + \sum_{a,b < c} \dot{u}_a(t) \dot{u}_b(t) \Gamma_{ab}^c(u_1, \dots, u_{c-1})(t) = 0.$$

We solve this system of equations recursively. Let $\gamma(t; \vec{u}^0, \vec{u}^1)$ be defined by

$$u_c(t) = u_c^0 + u_c^1 t - \int_0^t \int_0^s \sum_{a,b < c} \dot{u}_a(r) \dot{u}_b(r) \Gamma_{ab}^c(u_1, \dots, u_{c-1})(r) dr ds.$$

Then $\gamma(0; \vec{u}^0, \vec{u}^1)(0) = \vec{u}^0$ while $\dot{\gamma}(0; \vec{u}^0, \vec{u}^1)(0) = \vec{u}^1$. Thus every geodesic arises in this way so all geodesics extend for infinite time. Furthermore, given $P, Q \in \mathbb{R}^n$, there is a unique geodesic $\gamma = \gamma_{P,Q}$ so that $\gamma(0) = P$ and $\gamma(1) = Q$ where

$$u_c^0 = P_c, \quad u_c^1 = Q_c - P_c + \int_0^1 \int_0^s \sum_{a,b < c} \dot{u}_a(r) \dot{u}_b(r) \Gamma_{ab}^c(u_1, \dots, u_{c-1})(r) dr ds.$$

This shows that \exp_P is a diffeomorphism from $T_P \mathbb{R}^n$ to \mathbb{R}^n . \square

3. THE CURVATURE OF $\mathcal{M}_{2p+6,f}$

In view of Equation (2.a), in computing curvatures and higher covariant derivatives, only derivatives of highest weight play a role; we never need to consider quadratic terms in Christoffel symbols. Thus the non-zero curvatures are:

$$R_{2p+6,f}(\partial_x, \partial_y, \partial_y, \partial_x) = (\partial_y)^2 F_{2p+6,f}, \quad \text{and} \quad R_{2p+6,f}(\partial_x, \partial_y, \partial_{z_i}, \partial_x) = (i+1)y^i.$$

Proof of Theorem 1.6. Let ξ_i be arbitrary tangent vectors. Then:

$$\begin{aligned} \text{Range}\{\mathcal{R}_{2p+6,f}(\xi_1, \xi_2)\} &\subset \text{Span}_{C^\infty}\{\partial_{\bar{x}}, \partial_{\bar{y}}, \partial_{\bar{z}_0}, \dots, \partial_{\bar{z}_p}\}, \quad \text{and} \\ \text{Span}\{\partial_{\bar{x}}, \partial_{\bar{y}}, \partial_{\bar{z}_0}, \dots, \partial_{\bar{z}_p}\} &\subset \text{Ker}\{\mathcal{R}_{2p+6,f}(\xi_1, \xi_2)\}. \end{aligned}$$

Thus $\mathcal{R}_{2p+6,f}(\xi_1, \xi_2)\mathcal{R}_{2p+6,f}(\xi_3, \xi_4) = 0$ so $J_{2p+6,f}(\xi)^2 = 0$ and $\mathcal{R}_{2p+6,f}(\pi)^2 = 0$ for any tangent vector ξ and any non-degenerate 2-plane π . Consequently, $J_{2p+6,f}(\xi)$ and $\mathcal{R}_{2p+6,f}(\pi)$ have only the eigenvalue 0. \square

Similarly, the non-zero entries in $\nabla^k R$ for any $k \geq 0$ are given by:

$$\begin{aligned} \nabla^k R_{2p+6,f}(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y, \dots, \partial_y) &= (\partial_y)^{k+2} F_{2p+6,f}, \\ \nabla^k R_{2p+6,f}(\partial_x, \partial_y, \partial_{z_i}, \partial_x; \partial_y, \dots, \partial_y) &= \partial_{z_i} (\partial_y)^{k+1} F_{2p+6,f}, \quad \text{and} \\ \nabla^k R_{2p+6,f}(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y, \dots, \partial_{z_i}, \dots, \partial_y) &= \partial_{z_i} (\partial_y)^{k+1} F_{2p+6,f}. \end{aligned}$$

Proof of Theorem 1.4 (1). We may decompose $T\mathbb{R}^{2p+6} = \mathcal{V} \oplus \bar{\mathcal{V}}$ where

$$\begin{aligned} \mathcal{V} &:= \text{Span}\{\partial_x + \tfrac{1}{2}g_{2p+6,f}(\partial_x, \partial_x)\partial_{\bar{x}}, \partial_y, \partial_{z_0}, \dots, \partial_{z_p}\}, \quad \text{and} \\ \bar{\mathcal{V}} &:= \text{Span}\{\partial_{\bar{x}}, \partial_{\bar{y}}, \partial_{\bar{z}_0}, \dots, \partial_{\bar{z}_p}\}. \end{aligned}$$

Let π_1 denote projection on the first factor. There are tensors $A^k \in \otimes^{k+4}\mathcal{V}^*$ so that $\pi_1^* A^k = \nabla^k R$. Since \mathcal{V} is a totally isotropic subspace, this shows all scalar invariants formed using the Weyl calculus vanish. \square

4. A MODEL FOR $\mathcal{M}_{2p+6,f}$

We can now make a crucial observation. We have

$$(4.a) \quad \nabla^k R_{2p+6,f}(\partial_x, \partial_y, \partial_{z_i}, \partial_x; \partial_y, \dots, \partial_y) = \begin{cases} 0 & \text{if } i < k, \\ (k+1)! & \text{if } i = k. \end{cases}$$

Proof of Theorem 1.3 (3,4). We shall exploit the upper triangular form of Equation (4.a). Let $a^i(y, \bar{z})$ and $b_i^j(y, \bar{z})$ be smooth functions to be chosen presently. Set

$$X = \partial_x - \tfrac{1}{2}g_{2p+6,f}(\partial_x, \partial_x)\partial_{\bar{x}}, \quad Y = \partial_y + \sum_j a^j \partial_{z_j}, \quad \text{and} \quad Z_i = \sum_j b_i^j \partial_{z_j}.$$

Assume the matrix (b_i^j) is invertible; let (\hat{b}_i^j) be the inverse matrix. Set dually

$$\bar{X} = \partial_{\bar{x}}, \quad \bar{Y} = \partial_{\bar{y}}, \quad \text{and} \quad \bar{Z}_i = -\sum_j a^j \hat{b}_j^i \partial_{\bar{y}} + \sum_j \hat{b}_j^i \partial_{\bar{z}_j}.$$

This is then a hyperbolic basis, i.e. the first relation of Equation (1.a) holds.

We shall assume the matrix b_i^j is triangular:

$$Z_i = \sum_{j \leq i} b_i^j \partial_{z_j}.$$

The relation $\nabla^k R(X, Y, Y, X; Y, \dots, Y) = 0$ for $0 \leq k \leq p$ leads to the equations:

$$\begin{aligned} 0 &= \nabla^p R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y, \dots) + (p+1)a^p R(\partial_x, \partial_y, \partial_{z_p}, \partial_x; \partial_y, \dots), \\ 0 &= \nabla^{p-1} R(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y, \dots) + p \sum_{p-1 \leq i \leq p} a^i R(\partial_x, \partial_y, \partial_{z_i}, \partial_x; \partial_y, \dots), \quad \dots \\ 0 &= R(\partial_x, \partial_y, \partial_y, \partial_x) + \sum_{0 \leq i \leq p} a^i R(\partial_x, \partial_y, \partial_{z_i}, \partial_x). \end{aligned}$$

By Equation (4.a), $\nabla^k R(\partial_x, \partial_y, \partial_{z_k}, \partial_x; \partial_y, \dots) \neq 0$ and thus this triangular system of equations determines the coefficients a^i uniquely.

Similarly, the relations $\nabla^k R(X, Y, Z_j, X; Y, \dots) = \delta_{jk}$ leads to the equations:

$$\begin{aligned} 1 &= b_p^p \nabla^p R(\partial_x, \partial_y, \partial_{z_p}, \partial_x; \partial_y, \dots), \\ 1 &= b_{p-1}^{p-1} \nabla^p R(\partial_x, \partial_y, \partial_{z_{p-1}}, \partial_x; \partial_y, \dots), \\ 0 &= \sum_{p-1 \leq i \leq p} b_p^i \nabla^{p-1} R(\partial_x, \partial_y, \partial_{z_i}, \partial_x; \partial_y, \dots), \quad \dots, \\ 1 &= b_0^0 R(\partial_x, \partial_y, \partial_{z_0}, \partial_x), \\ 0 &= \sum_{0 \leq i \leq 1} b_1^i \nabla^{p-1} R(\partial_x, \partial_y, \partial_{z_i}, \partial_x; \partial_y, \dots), \\ 0 &= \sum_{0 \leq i \leq p} b_p^i R(\partial_x, \partial_y, \partial_{z_i}, \partial_x). \end{aligned}$$

This system of equations is triangular. First solve for b_p^p , then for $\{b_{p-1}^{p-1}, b_p^{p-1}\}$, and finally for $\{b_0^0, \dots, b_p^0\}$. Again, the fact that $\nabla^k R(\partial_x, \partial_y, \partial_{z_k}, \partial_y; \partial_y, \dots) \neq 0$ is crucial.

If $k > p$, then the only non-zero component of $\nabla^k R$ is given by

$$\nabla^k R_{2p+6,f}(\partial_x, \partial_y, \partial_y, \partial_x; \partial_y \dots \partial_y) = f^{(k+2)}(y).$$

There is still a bit of freedom left in the choice of basis. Let ε_0 and ε_1 be non-zero functions. We set

$$\begin{aligned} X^1 &= \varepsilon_0 X, & Y^1 &= \varepsilon_1 Y, & Z_0^1 &= \varepsilon_0^{-2} \varepsilon_1^{-1} Z_0, & \dots, & Z_p^1 &= \varepsilon_0^{-2} \varepsilon_1^{-p-1} Z_p, \\ \bar{X}^1 &= \varepsilon_0^{-1} \bar{X}, & \bar{Y}^1 &= \varepsilon_1^{-1} \bar{Y}, & \bar{Z}_0^1 &= \varepsilon_0^2 \varepsilon_1 \bar{Z}_0, & \dots, & \bar{Z}_p^1 &= \varepsilon_0^2 \varepsilon_1^{p+1} \bar{Z}_p. \end{aligned}$$

The normalizations of Equation (1.a) are preserved for $\{g_{2p+6,f}, R, \dots, \nabla^p R\}$. Also,

$$\begin{aligned} \nabla^{p+1} R_{2p+6,f}(X^1, Y^1, Y^1, X^1; Y^1 \dots Y^1) &= \varepsilon_0^2 \varepsilon_1^{p+3} f^{(p+3)}, \\ \nabla^{p+2} R_{2p+6,f}(X^1, Y^1, Y^1, X^1; Y^1 \dots Y^1) &= \varepsilon_0^2 \varepsilon_1^{p+4} f^{(p+4)}. \end{aligned}$$

As $f^{(p+3)} > 0$ and $f^{(p+4)} > 0$, we may set

$$\varepsilon_1 := \frac{f^{(p+3)}}{f^{(p+4)}} \quad \text{and} \quad \varepsilon_0 := \frac{1}{\{\varepsilon_1^{p+3} f^{(p+3)}\}^{\frac{1}{2}}}.$$

This shows that \mathcal{U}_{2p+6}^{p+2} is a $p+2$ model for $\mathcal{M}_{2p+6,f}$. \square

Proof of Theorem 1.5 (1). Suppose we set $f(y) = e^y$, $\varepsilon_0 = e^{-y/2}$ and $\varepsilon_1 = 1$. Then $\nabla^i R_{2p+6,f}(X^1, Y^1, Y^1, X^1; Y^1 \dots Y^1) = 1$ for any i . Consequently \mathcal{M}_{2p+6}^1 is a simply connected complete k -curvature homogeneous manifold for any k . Theorem 1.2 now implies \mathcal{M}_{2p+6}^1 is homogeneous. \square

Note that the full strength of Theorem 1.2 is not necessary. Results of Belger and Kowalski [1] show an analytic pseudo-Riemannian manifold which is k -curvature homogeneous for all k is locally homogeneous; in our setting the exponential coordinates are analytic diffeomorphisms so the qualifier ‘local’ can be removed.

5. A LOCAL INVARIANT

Let $k \geq p+1$. Define a generalization of the classical Jacobi operator by setting

$$J_{k,2p+6,f}(Y) : X \rightarrow \nabla_{Y, \dots, Y}^k R_{2p+6,f}(X, Y)Y.$$

Expand $X = a\partial_x + b\partial_y$ and $Y = c\partial_x + d\partial_y$. Then

$$J_{k,2p+6,f}(Y)X = (ad - bc)d^k f^{(k+2)}(d\partial_{\bar{x}} - c\partial_{\bar{y}}).$$

Proof of Theorem 1.4 (2). Choose $\{Y, X\}$ so $J_{p+1,2p+6,f}(Y)X \neq 0$. Then necessarily $d \neq 0$ and $(ad - bc) \neq 0$. Let h be **any** Riemannian metric on $\mathcal{M}_{2p+6,f}$;

$$\begin{aligned} & \frac{h(J_{p+1,2p+6,f}(Y)X, J_{p+3,2p+6,f}(Y)X)}{h(J_{p+2,2p+6,f}(Y)X, J_{p+2,2p+6,f}(Y)X)} \\ &= \frac{(ad - bc)^2 d^{2p+4} f^{(p+3)} f^{(p+5)} h(d\partial_{\bar{x}} - c\partial_{\bar{y}}, d\partial_{\bar{x}} - c\partial_{\bar{y}})}{(ad - bc)^2 d^{2p+4} f^{(p+4)} f^{(p+4)} h(d\partial_{\bar{x}} - c\partial_{\bar{y}}, d\partial_{\bar{x}} - c\partial_{\bar{y}})} \\ &= \alpha_{2p+6,f}. \end{aligned}$$

Thus $\alpha_{2p+6,f}$ is an affine invariant of $\{\nabla^{p+1}\mathcal{R}, \nabla^{p+2}\mathcal{R}, \nabla^{p+3}\mathcal{R}\}$. \square

Proof of Theorem 1.5 (2,3). If we set $f = e^y + e^{2y}$, then $\alpha_{2p+6,f}$ is not locally constant so \mathcal{M}_{2p+6}^2 is not locally $p+3$ -affine curvature homogeneous. It is, however, $p+2$ -curvature modeled on \mathcal{M}_{2p+6}^1 . \square

6. IRREDUCIBILITY

We restrict to the case $p = 0$. Set $f = 0$ to define $\mathcal{M}_{6,0}$. The discussion in Section 2 then yields that $\mathcal{M}_{6,0}$ is complete. The computations of Section 3 show $\nabla R_{g_{6,0}} = 0$ so $\mathcal{M}_{6,0}$ is a symmetric space. Furthermore the discussion of Section 4 shows that \mathcal{U}_6^0 is a 0-model for $\mathcal{M}_{6,0}$. Thus $\mathcal{M}_{6,0}$ is a 0-model for $\mathcal{M}_{6,f}$. We complete the proof of Theorem 1.7 by showing that \mathcal{U}_6^0 is irreducible as the other assertions then follow.

Let $Z = Z_0$ and $\bar{Z} = \bar{Z}_0$. Let $\mathbb{R}^3 = \text{Span}\{X, Y, Z\}$. We consider an affine model $\mathcal{V} = (\mathbb{R}^3, B)$ where $B \in \otimes^4(\mathbb{R}^3)^*$ is defined by

$$B(X, Y, Z, X) = 1.$$

Lemma 6.1. *The affine model \mathcal{V} is irreducible.*

Proof. Suppose a non-trivial decomposition $\mathbb{R}^3 = V_1 \oplus V_2$ induces a corresponding decomposition $B = B_1 \oplus B_2$. Assume the notation chosen so $\dim(V_1) = 2$ and $\dim(V_2) = 1$. Let $0 \neq \xi \in V_2$. Since $\dim(V_2) = 1$, $B_2 = 0$ so $B(\eta_1, \eta_2, \eta_3, \xi) = 0$ for all $\eta_i \in \mathbb{R}^3$. We expand $\xi = aX + bY + cZ$. We then have

$$a = B(\xi, Y, Z, X) = 0, \quad b = B(X, \xi, Z, X) = 0, \quad \text{and} \quad c = B(X, Y, \xi, X) = 0.$$

Thus $\xi = 0$ which is false. This contradiction proves the Lemma. \square

Let π be the natural projection from \mathbb{R}^6 to $W := \mathbb{R}^6/\mathcal{K}$ where

$$\mathcal{K} := \{\xi \in \mathbb{R}^6 : A_6^0(\eta_1, \eta_2, \eta_3, \xi) = 0 \quad \forall \quad \eta_i \in \mathbb{R}^3\} = \text{Span}\{\bar{X}, \bar{Y}, \bar{Z}\}.$$

We suppose \mathcal{U}_6^0 is reducible and argue for a contradiction. Let $\mathbb{R}^6 = V_1 \oplus V_2$ be a non-trivial decomposition with a corresponding decomposition

$$(6.a) \quad g_{6,0} = g_{6,0,1} \oplus g_{6,0,2} \quad \text{and} \quad A_6^0 = A_{6,1}^0 \oplus A_{6,2}^0.$$

This also induces a decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$. We set $W_i := V_i/\mathcal{K}_i$ to decompose $W = W_1 \oplus W_2$ and $B = B_1 \oplus B_2$. By Lemma 6.1, this decomposition is trivial; we choose the notation so $W_2 = \{0\}$ and hence $V_2 \subset \mathcal{K}_2 \subset \mathcal{K}$. Since \mathcal{K} is a null subspace, $g_{6,0,2}$ is trivial. This is a contradiction as $g_{6,0} = g_{6,0,1} \oplus g_{6,0,2}$ and $g_{6,0}$ is non-singular. This contradiction completes the proof of Theorem 1.7. \square

ACKNOWLEDGMENTS

Research of P. Gilkey partially supported by the Max Planck Institute in the Mathematical Sciences (Leipzig). Research of S. Nikčević partially supported by MM 1646 (Srbija). We are grateful to Professors O. Kowalski and L. Vanhecke for introducing us to this area in the first instance and to Prof. García-Río for suggesting this as a fruitful area of inquiry. We also acknowledge with pleasure helpful conversations with Dra. S. López Ornat.

REFERENCES

- [1] M. Belger and O. Kowalski, Riemannian metrics with the prescribed curvature tensor and all its covariant derivatives at one point, *Math. Nachr.* **168** (1994), 209–225.
- [2] E. Boeckx, L. Vanhecke, and O. Kowalski, **Riemannian manifolds of conullity two**, World Scientific (1996).
- [3] P. Bueken and M. Djorić, Three-dimensional Lorentz metrics and curvature homogeneity of order one, *Ann. Global Anal. Geom.* **18** (2000), 85–103.
- [4] P. Bueken, and L. Vanhecke, Examples of curvature homogeneous Lorentz metrics, *Classical Quantum Gravity* **14** (1997), L93–L96.

- [5] M. Cahen, J. Leroy, M. Parker, F. Tricerri, and L. Vanhecke, Lorentz manifolds modeled on a Lorentz symmetric space, *J. Geom. Phys.* **7** (1990), 571–581.
- [6] G. Calvaruso, R. A. Marinosci, and D. Perrone, Three-dimensional curvature homogeneous hypersurfaces, *Arch. Math. (Brno)* **36** (2000), 269–278.
- [7] C. Dunn, Thesis University of Oregon (2005).
- [8] D. Ferus, H. Karcher, H. Münzner, Cliffordalgebren und neue isoparametrische Hyperflächen, *Math. Z.* **177** (1981), 479–502.
- [9] E. García-Río, D. Kupeli, M. E. Vázquez-Abal, and R. Vázquez-Lorenzo, Affine Osserman connections and their Riemann extensions. *Differential Geom. Appl.* **11** (1999), 145–153.
- [10] E. García-Río, D. Kupeli, and R. Vázquez-Lorenzo, **Osserman Manifolds in Semi-Riemannian Geometry**, Lecture Notes in Mathematics, 1777. Springer-Verlag, Berlin, 2002. ISBN: 3-540-43144-6.
- [11] P. Gilkey, **Geometric properties of natural operators defined by the Riemann curvature tensor**, World Scientific Publishing Co., Inc., River Edge, N. J., 2001.
- [12] P. Gilkey and S. Nikčević, Complete curvature homogeneous pseudo-Riemannian manifolds, math.DG/0402282.
- [13] —, Curvature homogeneous spacelike Jordan Osserman pseudo-Riemannian manifolds, to appear *Classical Quantum Gravity*, math.DG/0310024.
- [14] M. Gromov, **Partial differential relations**, *Ergeb. Math. Grenzgeb.* 3. Folge, Band 9, Springer-Verlag (1986).
- [15] O. Kowalski, B. Opozda, and Z. Vlášek, Curvature homogeneity of affine connections on two-dimensional manifolds, *Colloq. Math.* **81** (1999), 123–139.
- [16] —, A classification of locally homogeneous affine connections with skew-symmetric Ricci tensor on 2 dimensional manifolds, *Monatsh. Math.* **130** (2000), 109–125.
- [17] —, A classification of locally homogeneous connections on 2-dimensional manifolds via group-theoretical approach, *Central European Journal of Mathematics* (2004) **2**, 87–102.
- [18] O. Kowalski and F. Prüfer, Curvature tensors in dimension four which do not belong to any curvature homogeneous space, *Arch. Math. (Brno)* **30** (1994), 45–57.
- [19] O. Kowalski, F. Tricerri, and L. Vanhecke, Curvature homogeneous Riemannian manifolds *J. Math. Pures Appl.* **71** (1992), 471–501.
- [20] —, Curvature homogeneous spaces with a solvable Lie group as homogeneous model, *J. Math. Soc. Japan* **44** (1992), 461–484.
- [21] B. Opozda, On curvature homogeneous and locally homogeneous affine connections, *Proc. Amer. Math. Soc.* **124** (1996), 1889–1893.
- [22] B. Opozda, Affine versions of Singer’s theorem on locally homogeneous spaces, *Ann. Global Anal. Geom.* **15** (1997), 187–199.
- [23] V. Pravda, A. Pravdová, A. Coley, and R. Milson, All spacetimes with vanishing curvature invariants, *Classical Quantum Gravity* **19** (2002), 6213–6236.
- [24] F. Prüfer, F. Tricerri, and L. Vanhecke, Curvature invariants, differential operators and local homogeneity, *Trans. Am. Math. Soc.* **348** (1996), 4643–4652.
- [25] F. Podesta and A. Spiro, Introduzione ai Gruppi di Trasformazioni, Volume of the Preprint Series of the Mathematics Department “V. Volterra” of the University of Ancona, Via delle Brecce Bianche, Ancona, ITALY (1996).
- [26] I. M. Singer, Infinitesimally homogeneous spaces, *Commun. Pure Appl. Math.* **13** (1960), 685–697.
- [27] K. Sekigawa, H. Suga, and L. Vanhecke, Four-dimensional curvature homogeneous spaces, *Commentat. Math. Univ. Carol.* **33** (1992), 261–268.
- [28] K. Sekigawa, H. Suga, and L. Vanhecke, Curvature homogeneity for four-dimensional manifolds, *J. Korean Math. Soc.* **32** (1995), 93–101.
- [29] H. Takagi, On curvature homogeneity of Riemannian manifolds, *Tôhoku Math. J.* **26** (1974), 581–585.
- [30] A. Tomassini, Curvature homogeneous metrics on principal fibre bundles, *Ann. Mat. Pura Appl.* **172** (1997), 287–295.
- [31] F. Tricerri, Riemannian manifolds with the same curvature as a homogeneous space, and a conjecture of Gromov, **Geometry Conference** (Parma, 1988). Riv. Mat. Univ. Parma (4) **14** (1988), 91–104.
- [32] K. Tsukada, Curvature homogeneous hypersurfaces immersed in a real space form, *Tohoku Math. J.* **40** (1988), 221–244.
- [33] F. Tricerri and L. Vanhecke, Variétés riemanniennes dont le tenseur de courbure est celui d’un espace symétrique riemannien irréductible, *C. R. Acad. Sci., Paris, Sér. I* **302** (1986), 233–235.
- [34] L. Vanhecke, Curvature homogeneity and related problems, **Proceedings of the Workshop on Recent Topics in Differential Geometry** (Puerto de la Cruz, 1990), 103–122, Informes, 32, Univ. La Laguna, La Laguna, 1991.

[35] K. Yamato, Algebraic Riemann manifolds, *Nagoya Math. J.* **115** (1989), 87–104.

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